VOLUME 76 SEPARATE No. 14

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

APRIL, 1950

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

MATHEMATICAL ANALYSIS OF AN AERIAL SURVEY

By Lo-Ho

SURVEYING AND MAPPING DIVISION

Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

PRICE $0.50 PER COPY

Published at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York 18, N. Y. Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

MATHEMATICAL ANALYSIS OF AN AERIAL SURVEY

By Lo-Ho[1]

Synopsis

Aerial photographic methods are now widely used in topographic surveying, and in recent years efforts have been made to place them under mathematical control by analytical procedures of computation. In this field Earl Church,[2] Assoc. M. ASCE, and P. H. Underwood,[3] M. ASCE, are the leaders. The purpose of this paper is to present an analytical solution based on the direction-cosine principles of space geometry. The underlying mathematical principles are first developed and then applied to aerial survey problems. The methods of analysis and adjustment and the determination of the effect of the atmospheric refraction are new.

Data Used

The data immediately available in an aerial survey are the projections of the ground points on the negative plate, as transmitted through the objective lens; but the angles at the front nodal point, subtended by the separations of the objects at the instant of exposure, will be used in the following analysis. Furthermore, the positions of three or more ground points are also required, and these will be provided by the ground survey. The space angles at the front nodal point are assumed equal to their corresponding angles at the rear nodal point. The latter may be determined if the perpendicular distance from the rear nodal point to the plane of the negative plate, and the coordinates of the base of this perpendicular with respect to the fiducial axes on the negative plate, can be determined. An analysis of the geometrical relation between the rear nodal point and the fiducial axes is a part of this paper. The effect of atmospheric refraction is also analyzed; but the measurement of the coordinates of points in the negative by a comparator or other special instrument, and the

Note.—Written comments are invited for publication; the last discussion should be submitted by September 1, 1950.

[1] Prof. of Surveying and Geodesy, Tangshan Eng. College, Tangshan, North China.

[2] "Manual of Photogrammetry," Pitman Pub. Corp., New York, N. Y., and Chicago, Ill., 1944, p. 536.

[3] "Space Resection Problems in Photogrammetry," by P. H. Underwood, *Transactions*, ASCE, Vol. 112, 1947, p. 921.

calibration of the camera to determine values of distortion, are not considered in the present paper.

Mathematical Principles

For convenience of reference, a summary of the mathematical principles directly involved in the following analysis will be presented first. Some well-known formulas are given without proof; but, for those not commonly known, the method of derivation will be outlined.

The method of adjustment described consists largely of the determination of corrections to be applied to computed or observed values. When the true values are connected by one or several conditional equations, the approximate relations between the corrections may be obtained by differentiation of such conditional equations, since the corrections to be applied are quite small in comparison with the values to be corrected. The resulting formulas will all be linear equations containing the corrections as unknowns and the differentials as exponents whose values may be found from the known values of the quantities. This general method will be used to derive many of the required equations.

Reference Axes.—The normal line through point O on the sphere OSWNE, Fig. 1, is adopted as the Z-axis; the north–south line through O, in the tangential plane of the sphere at point O, is the Y-axis; and the east–west line is the X-axis. The positive directions are all indicated by arrowheads as shown. The center of the sphere is designated point C, and its radius by R.

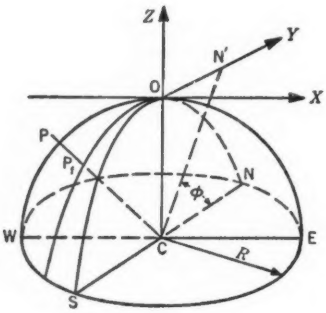

Fig. 1

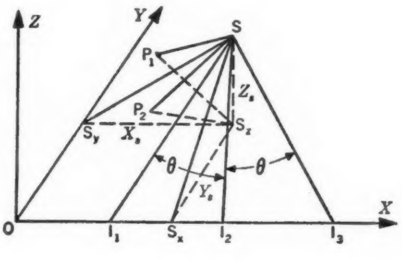

Fig. 2

Normal Distance of a Point to the Sphere.—Let the coordinates of point P be X_p, Y_p, and Z_p, and let a normal line from point P on the sphere be drawn to point P_f as shown in Fig. 1. Then

$$\overline{P_fP} = [X^2_p + Y^2_p + (Z_p + R)^2]^{\frac{1}{2}} - R$$

$$= Z_p + \frac{X^2_p + Y^2_p}{2\,(Z_p + R)} - \frac{(X^2_p + Y^2_p)^2}{8\,(Z_p + R)^3} + \cdots \qquad (1)$$

which may be used to determine the elevation of a point defined by coordinate distances.

Length of the Arc OP_f.—The curved distance, OP_f, on the sphere is given by

$$\overline{\mathrm{OP}_f} = R \sin^{-1} \frac{(X^2{}_p + Y^2{}_p)^{\frac{1}{2}}}{R + \overline{\mathrm{P}_f\mathrm{P}}} \qquad (2)$$

Eq. 2 may be used to determine the length of line OP at sea level.

Equation of a Normal Line.—The equation of the radial line PC is

$$\frac{X}{X_p} = \frac{Y}{Y_p} = \frac{Z + R}{Z_p + R} \qquad (3)$$

which may be used to determine the equation of the plumb line from a point having known coordinates.

Coordinates of a Point from Its Elevation and Distance.—Let the azimuth of line OP, measured from the north point in clockwise direction be α; its curved distance OP_f be s; and its elevation $\mathrm{P}_f\mathrm{P}$ be h. Then

$$X_p = (R + h) \sin \frac{s}{R} \sin \alpha \qquad (4a)$$

$$Y_p = (R + h) \sin \frac{s}{R} \cos \alpha \qquad (4b)$$

and

$$Z_p = (R + h) \cos \frac{s}{R} - R \qquad (4c)$$

Eqs. 4 may be used to determine the coordinates of points located by a ground survey.

Azimuth Angle of a Line.—Let N′ be a point on the Y-axis and let the vertical angle of the line CN′ be ϕ as shown in Fig. 1. Then the coordinates of N′ are 0, $R \cot \phi$, and 0; and the equation of the plane containing points N′, C, and P is

$$\begin{vmatrix} X & Y & Z & 1 \\ 0 & R \cot \phi & 0 & 1 \\ 0 & 0 & -R & 1 \\ X_p & Y_p & Z_p & 1 \end{vmatrix} = 0 \qquad (5)$$

Let P_1 be another point with coordinates X_1, Y_1, and Z_1. The equation of the plane containing points C, P, and P_1 is found to be

$$\begin{vmatrix} X & Y & X & 1 \\ 0 & 0 & -R & 1 \\ X_p & Y_p & Z_p & 1 \\ X_1 & Y_1 & Z_1 & 1 \end{vmatrix} = 0 \qquad (6)$$

Eq. 5 can be used to determine the meridian plane through a given point; Eq. 6 gives the vertical plane through two given points; and the angle between the two planes is the azimuth of line PP_1, which may be determined from the equations of the two planes by the usual method.

Space Angles at a Point.—The coordinates OS_x, OS_y, and S_zS of point S, Fig. 2, with respect to the rectangular axes are X_s, Y_s, and Z_s, respectively. Similarly the coordinates of points P_1 and P_2 in the (XY)-plane are denoted by X_1, Y_1, X_2, and Y_2, respectively. Angle P_1SP_2 may be determined from these coordinates thus:

$$\overline{S_zP_1} = \sqrt{(X_1 - X_s)^2 + (Y_1 - Y_s)^2} \quad \text{(7a)}$$

$$\overline{S_zP_2} = \sqrt{(X_2 - X_s)^2 + (Y_2 - Y_s)^2} \quad \text{(7b)}$$

$$\overline{P_1P_2} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \quad \text{(7c)}$$

$$\overline{SP_1} = \sqrt{Z^2_s + (\overline{S_zP_1})^2} \quad \text{(7d)}$$

$$\overline{SP_2} = \sqrt{Z^2_s + (\overline{S_zP_2})^2} \quad \text{(7e)}$$

and

$$\cos \angle \overline{P_1SP_2} = \frac{(\overline{SP_1})^2 + (\overline{SP_2})^2 - (\overline{P_1P_2})^2}{2\,\overline{SP_1}\,\overline{SP_2}} \quad \text{(7f)}$$

Eqs. 7 can be used to determine the space angle at an exposure station. Lines SS_x and SS_y are perpendicular to axes OX and OY, respectively.

Coordinates in Terms of Axial Distances.—Let points I_1, I_2, and I_3, Fig. 2, be points on the X-axis forming equal space angles I_1SI_2 and I_2SI_3 to be denoted by θ; and let the angle S_xSI_2 be denoted by E_x. Then, from the relation, $\frac{\overline{SI_3}}{\overline{SI_1}} = \frac{\overline{I_2I_3}}{\overline{I_1I_2}} = \frac{\cos(\theta - E_x)}{\cos(\theta + E_x)}$, it may be found that

$$\tan E_x = \frac{\overline{I_2I_3} - \overline{I_1I_2}}{\overline{I_1I_3}} \cot \theta \quad \text{(8)}$$

Since $\overline{S_xI_2} = \overline{SS_x} \tan E_x$ and $\overline{S_xI_2} + \overline{I_2I_3} = \overline{SS_x} \tan(\theta + E_x)$, it may be found that

$$\overline{SS_x} = \frac{\overline{I_2I_3}}{\tan(\theta + E_x) - \tan E_x} \quad \text{(9)}$$

and

$$\overline{S_xI_2} = \frac{\overline{I_2I_3} \tan E_x}{\tan(\theta + E_x) - \tan E_x} \quad \text{(10)}$$

From Fig. 2, it is evident that

$$X_s = \overline{OI_2} - \overline{S_xI_2} \quad \text{(11)}$$

and

$$Z_s = \sqrt{(\overline{SS_x})^2 - Y^2_s} = \sqrt{(\overline{SS_y})^2 - X^2_s} \quad \text{(12)}$$

Formulas similar to Eqs. 8 to 11 may be derived, if three points on the Y-axis subtend equal space angles at point S. Eqs. 8 through 12 can be used to determine the perpendicular distance from the rear node of a lens to the plane of film emulsion, and the coordinates of the foot can be determined with respect to the fiducial axes.

Equation of Straight Line.—The coordinates of any point P_n will be denoted by X_n, Y_n, and Z_n and their small corrections by x_n, y_n, and z_n. Direction cosines of a line in the directions of X, Y, and Z will be denoted by L, M, and N and their small corrections by l, m, and n, respectively. The equation of line P_1P_2 is

$$\frac{X - X_1}{L} = \frac{Y - Y_1}{M} = \frac{Z - Z_1}{N} \quad \text{(13)}$$

in which

$$L = \frac{X_2 - X_1}{r} \quad \text{(14a)}$$

$$M = \frac{Y_2 - Y_1}{r} \quad \text{(14b)}$$

and

$$N = \frac{Z_2 - Z_1}{r} \quad \text{(14c)}$$

In Eqs. 14,

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \quad \text{(15)}$$

If small corrections x_1, y_1, and z_1 are added to the coordinates of point P_1, it can be found that

$$l = \frac{1}{r}[-(1 - L^2)\, x_1 + L\, M\, y_1 + L\, N\, z_1] \quad \text{(16a)}$$

$$m = \frac{1}{r}[L\, M\, x_1 - (1 - M^2)\, y_1 + M\, N\, z_1] \quad \text{(16b)}$$

and

$$n = \frac{1}{r}[L\, N\, x_1 + M\, N\, y_1 - (1 - N^2)\, z_1] \quad \text{(16c)}$$

Eq. 16c can be written

$$n = -\frac{L\, l + M\, m}{N} \quad \text{(17)}$$

Distance from a Point to a Line.—The perpendicular distance D_n from any point P_n to the line defined by Eq. 13 is given by

$$D^2_n = (X_1 - X_n)^2 + (Y_1 - Y_n)^2 + (Z_1 - Z_n)^2 - [L\,(X_1 - X_n) + M\,(Y_1 - Y_n) + N\,(Z_1 - Z_n)]^2 \quad \text{(18)}$$

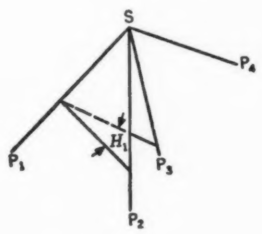

FIG. 3

Angle Between Two Intersecting Lines.—Let the direction cosines of two lines SP_1 and SP_2, Fig. 3, be denoted by L_1, M_1, N_1, L_2, M_2, and N_2, respectively. The angle A_{12} included by the two lines (P_1S and SP_2, Fig. 3) is given by

$$\cos A_{12} = L_1\, L_2 + M_1\, M_2 + N_1\, N_2 \quad \text{(19a)}$$

The correction a_{12} (in radians) of angle A_{12} due to small corrections of the direction cosines is given by

$$a_{12} = -\operatorname{cosec} A_{12}\,(L_2\, l_1 + L_1\, l_2 + M_2\, m_1 + M_1\, m_2 + N_2\, n_1 + N_1\, n_2) \quad \text{(19b)}$$

If the corrections of the direction cosines are due to the corrections x, y, and z of the coordinates of point S alone, it may be found that

$$a_{12} = \frac{\operatorname{cosec} A_{12}}{r_1 r_2} \{[(r_1 L_1 + r_2 L_2) - (r_1 L_2 + r_2 L_1) \cos A_{12}] x$$
$$+ [(r_1 M_1 + r_2 M_2) - (r_1 M_2 + r_2 M_1) \cos A_{12}] y$$
$$+ [(r_1 N_1 + r_2 N_2) - (r_1 N_2 + r_2 N_1) \cos A_{12}] z\} \quad (20)$$

in which r_1 and r_2 are the original lengths of the lines SP_1 and SP_2, respectively.

Distance Between Two Lines.—The perpendicular distance between two nonintersecting lines, $\frac{X - X_1}{L_1} = \frac{Y - Y_1}{M_1} = \frac{Z - Z_1}{N_1}$ and $\frac{X - X_2}{L_2} = \frac{Y - Y_2}{M_2} = \frac{Z - Z_2}{N_2}$, is given by

$$d = \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}}{[(M_1 N_2 - M_2 N_1)^2 + (L_1 N_2 - L_2 N_1)^2 + (L_1 M_2 - L_2 M_1)^2]^{\frac{1}{2}}} \quad (21a)$$

The condition of intersection of the two lines is controlled by

$$\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix} = 0 \quad (21b)$$

Small corrections x_1, x_2, y_1, y_2, z_1, z_2, l_1, l_2, m_1, m_2, n_1, and n_2 to be added to the coordinates and the direction cosines of two nearly intersecting lines to make them intersect are connected by

$$\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix} + \begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & z_1 - z_2 \\ L_1 & M_1 & n_1 \\ L_2 & M_2 & n_2 \end{vmatrix}$$
$$+ \begin{vmatrix} X_1 - X_2 & y_1 - y_2 & Z_1 - Z_2 \\ L_1 & m_1 & N_1 \\ L_2 & m_2 & N_2 \end{vmatrix} + \begin{vmatrix} x_1 - x_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ l_1 & M_1 & N_1 \\ l_2 & M_2 & N_2 \end{vmatrix} = 0 \quad (22)$$

Eq. 22 was obtained by neglecting small terms involving powers of the corrections, which powers are higher than the first.

The coordinates X_m, Y_m, and Z_m of the middle point P_m of the line perpendicular to the foregoing two lines may be found to be

$$X_m = \tfrac{1}{2} (X_1 + X_2)$$
$$- \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ 2 L_1 L_2 & L_1 M_2 + L_2 M_1 & L_1 N_2 + L_2 N_1 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2 [(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2]} \quad (23a)$$

$$Y_m = \frac{1}{2}(Y_1 + Y_2) - \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 M_2 + L_2 M_1 & 2 M_1 M_2 & M_1 N_2 + M_2 N_1 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2\left[(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2\right]} \quad \text{(23b)}$$

and

$$Z_m = \frac{1}{2}(Z_1 + Z_2) - \frac{\begin{vmatrix} X_1 - X_2 & Y_1 - Y_2 & Z_1 - Z_2 \\ L_1 N_2 + L_2 N_1 & M_1 N_2 + M_2 N_1 & 2 N_1 N_2 \\ M_1 N_2 - M_2 N_1 & L_2 N_1 - L_1 N_2 & L_1 M_2 - L_2 M_1 \end{vmatrix}}{2\left[(M_1 N_2 - M_2 N_1)^2 + (L_2 N_1 - L_1 N_2)^2 + (L_1 M_2 - L_2 M_1)^2\right]} \quad \text{(23c)}$$

which may be used to determine the most probable point of intersection of two observed lines that should meet at a point if there were no errors of observation.

Three Concurrent Lines.—Let the space angles between the three concurrent lines SP_1, SP_2, and SP_3, as shown in Fig. 3, be denoted by A_{12}, A_{13}, and A_{23}, respectively; and let H_1 denote the dihedral angle between the two planes SP_1P_2 and SP_1P_3. By the law of cosines in spherical trigonometry,

$$\cos H_1 = \frac{\cos A_{23} - \cos A_{12} \cos A_{13}}{\sin A_{12} \sin A_{13}} \quad \text{(24)}$$

If H_1 remains unchanged, but A_{12} and A_{13} are corrected by $a_{12} = -\rho \tan A_{12}$ and $a_{13} = -\rho \tan A_{13}$, respectively, the correction to be applied to A_{23} is given by

$$a_{23} = \rho \operatorname{cosec} A_{23} \left(2 \cos A_{23} - \frac{\cos A_{13}}{\cos A_{12}} - \frac{\cos A_{12}}{\cos A_{13}}\right) \quad \text{(25)}$$

When the direction cosines L_1, M_1, N_1, L_2, M_2, and N_2 and the space angles A_{13} and A_{23} are known, the direction cosines of line SP_3 may be found by solving the equations:

$$\cos A_{13} = L_1 L_3 + M_1 M_3 + N_1 N_3 \quad \text{(26a)}$$

$$\cos A_{23} = L_2 L_3 + M_2 M_3 + N_2 N_3 \quad \text{(26b)}$$

and

$$L^2_3 + M^2_3 + N^2_3 = 1 \quad \text{(26c)}$$

From Eqs. 26*a* and 26*b*, the magnitudes L_3 and M_3 may be found in terms of N_3; and, by substitution in Eq. 26*c*, a quadratic is obtained which may be solved for the root N_3.

If points P_1 and P_2 are fixed in position, if point S is moved a little to a new position having $(X + x)$, $(Y + y)$, and $(Z + z)$ as its new coordinates, and if the space angles A_{13} and A_{23} are given small corrections a_{13} and a_{23}, respectively, the corrections l_1, m_1, n_1, l_2, m_2, and n_2 may be found in terms of x, y, and z by Eqs. 16 and l_3, m_3, and n_3 may be found in terms of x, y, and z through

the solution of the equations:

$$L_1 l_3 + M_1 m_3 + N_1 n_3 = -\sin A_{13} a_{13} - L_3 l_1 - M_3 m_1 - N_3 n_1 \quad (27a)$$

$$L_2 l_3 + M_2 m_3 + N_2 n_3 = -\sin A_{23} a_{23} - L_3 l_2 - M_3 m_2 - N_3 n_2 \quad (27b)$$

and

$$L_3 l_3 + M_3 m_3 + N_3 n_3 = 0 \quad (27c)$$

which are obtained by differentiating Eqs. 26.

Four Concurrent Lines.—Let the space angles between the four lines SP_1, SP_2, SP_3, and SP_4, in Fig. 3, be denoted by A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, and A_{34}. Since the dihedral angles between the planes SP_1P_2, SP_1P_3, and SP_1P_4 can be expressed in terms of the space angles by Eq. 24 and the sum or difference of two is equal to the third, by equating the cosine of the one to the cosine of the sum of the other two, it is found that

$$\cos A_{34} = \cos A_{13} \cos A_{14} + \frac{K_{123} K_{124}}{\sin^2 A_{12}} + \frac{(\cos A_{23} - \cos A_{12} \cos A_{13})(\cos A_{24} - \cos A_{12} \cos A_{14})}{\sin^2 A_{12}} \quad (28)$$

in which

$$K_{123} = (1 + 2 \cos A_{12} \cos A_{13} \cos A_{23} - \cos^2 A_{12} - \cos^2 A_{13} - \cos^2 A_{23})^{\frac{1}{2}} \quad (29a)$$

and

$$K_{124} = (1 + 2 \cos A_{12} \cos A_{14} \cos A_{24} - \cos A_{12} - \cos^2 A_{14} - \cos^2 A_{24})^{\frac{1}{2}} \quad (29b)$$

The second term on the right side of Eq. 28 will have a negative sign when points P_3 and P_4 are on the opposite sides of plane SP_1P_2. The small corrections of the six space angles are connected by

$$a_{34} = c_{12} a_{12} + c_{13} a_{13} + c_{14} a_{14} + c_{23} a_{23} + c_{24} a_{24} \quad (30)$$

in which the differential coefficients are

$$c_{12} = 2 \cot A_{12} \cot A_{34} - \frac{K_{124} (\cos A_{12} - \cos A_{13} \cos A_{23})}{K_{123} \sin A_{12} \sin A_{34}} - \frac{K_{123} (\cos A_{12} - \cos A_{14} \cos A_{24})}{K_{124} \sin A_{12} \sin A_{34}} - \frac{\cos A_{14} \cos A_{23} + \cos A_{13} \cos A_{24}}{\sin A_{12} \sin A_{34}} \quad (31a)$$

$$c_{13} = \frac{\sin A_{13} (\cos A_{14} - \cos A_{12} \cos A_{24})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{124} \sin A_{13} (\cos A_{13} - \cos A_{12} \cos A_{23})}{K_{123} \sin^2 A_{12} \sin A_{34}} \quad (31b)$$

$$c_{14} = \frac{\sin A_{14} (\cos A_{13} - \cos A_{12} \cos A_{23})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{123} \sin A_{14} (\cos A_{14} - \cos A_{12} \cos A_{24})}{K_{124} \sin^2 A_{12} \sin A_{34}} \quad \text{(31}c\text{)}$$

$$c_{23} = \frac{\sin A_{23} (\cos A_{24} - \cos A_{12} \cos A_{14})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{124} \sin A_{23} (\cos A_{23} - \cos A_{12} \cos A_{13})}{K_{123} \sin^2 A_{12} \sin A_{34}} \quad \text{(31}d\text{)}$$

and

$$c_{24} = \frac{\sin A_{24} (\cos A_{23} - \cos A_{12} \cos A_{13})}{\sin^2 A_{12} \sin A_{34}} - \frac{K_{123} \sin A_{24} (\cos A_{24} - \cos A_{12} \cos A_{14})}{K_{124} \sin^2 A_{12} \sin A_{34}} \quad \text{(31}e\text{)}$$

If the angles A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, and A_{34} are determined by the coordinates of points S, P_1, P_2, P_3, and P_4, the value of A_{34}, obtained from the triangle SP_3P_4, will be identical with that determined from Eq. 28.

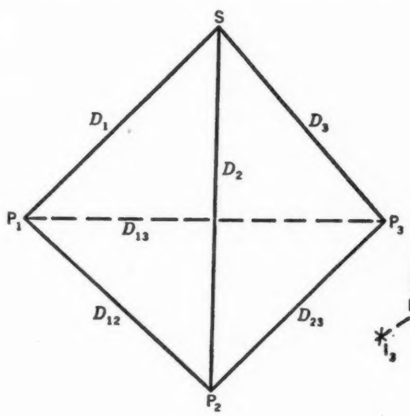

FIG. 4

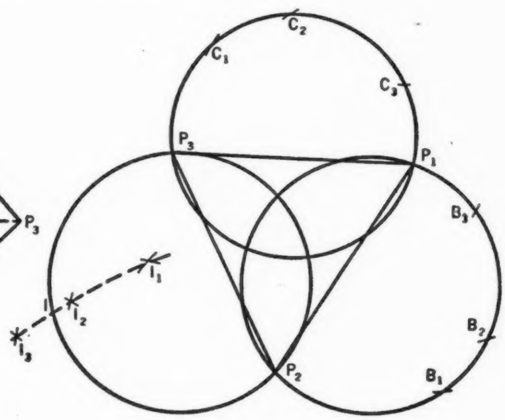

FIG. 5

If the conditional formula (Eq. 28) is satisfied, the direction cosines L_4, M_4, and N_4 of line SP_4 are determined from the values of the direction cosines in the other three lines by the simultaneous linear equations:

$$\cos A_{14} = L_1 L_4 + M_1 M_4 + N_1 N_4 \quad \text{(32}a\text{)}$$

$$\cos A_{24} = L_2 L_4 + M_2 M_4 + N_2 N_4 \quad \text{(32}b\text{)}$$

and

$$\cos A_{34} = L_3 L_4 + M_3 M_4 + N_3 N_4 \quad \text{(32}c\text{)}$$

Angles and Sides of a Tetrahedron.—The space angles of the tetrahedron at the vertex S, Fig. 4, are denoted by A_{12}, A_{13}, and A_{23}; and the lengths of the sides, SP_1, SP_2, SP_3, P_1P_2, P_1P_3, and P_2P_3, are denoted by D_1, D_2, D_3, D_{12}, D_{13},

and D_{23}, respectively. The relations between these angles and sides are

$$D^2_{12} = D^2_1 + D^2_2 - 2\,D_1\,D_2 \cos A_{12} \qquad \text{(33a)}$$

$$D^2_{13} = D^2_1 + D^2_3 - 2\,D_1\,D_3 \cos A_{13} \qquad \text{(33b)}$$

and

$$D^2_{23} = D^2_2 + D^2_3 - 2\,D_2\,D_3 \cos A_{23} \qquad \text{(33c)}$$

which are usually used to determine D_1, D_2, and D_3 from the known values of A_{12}, A_{13}, A_{23}, D_{12}, D_{13}, and D_{23}. The small corrections d_1, d_2, and d_3 of D_1, D_2, and D_3, respectively, are connected by

$$d_2 = \frac{D_1 - D_2 \cos A_{12}}{D_1 \cos A_{12} - D_2}\,d_1 \qquad \text{(34a)}$$

$$d_3 = \frac{D_1 - D_3 \cos A_{13}}{D_1 \cos A_{13} - D_3}\,d_1 \qquad \text{(34b)}$$

and

$$d_2 = \frac{D_3 - D_2 \cos A_{23}}{D_3 \cos A_{23} - D_2}\,d_3 \qquad \text{(34c)}$$

Combining Eqs. 34*b* and 34*c*,

$$d_2 = \frac{D_1 - D_3 \cos A_{13}}{D_1 \cos A_{13} - D_3} \times \frac{D_3 - D_2 \cos A_{23}}{D_3 \cos A_{23} - D_2}\,d_1 \qquad \text{(35)}$$

Geometrical Solution of a Tetrahedron.—To determine D_1, D_2, and D_3 geometrically, first construct a plane triangle $P_1P_2P_3$ to scale and draw circular arcs on its three sides to contain the vertex angles A_{12}, A_{13}, and A_{23}, respectively, as shown in Fig. 5. Then with an estimated value of D_1 as radius and point P_1 as a center draw a circular arc to intersect the circles on either side of point P_1 at points B_1 and C_1 as shown. Next, with P_2B_1 ($= P_3C_1$) as radius, draw the intersecting arcs at I_1, Fig. 5, using points P_2 and P_3, respectively, as centers. If point I_1 falls within the third circle as shown, the assumed value of D_1 is probably too long.

Use a revised value of D_1 to repeat the same operation, thus determining another point I_2, and, if necessary, a third point I_3. Let the smooth curve through points I_1, I_2, and I_3 intersect the outer arc of the third circle at point I, Fig. 5. The distances P_2I and P_3I are the required values of D_2 and D_3 respectively. With line P_2I representing the value of D_2, it is an easy matter to determine D_1. The foregoing method can be used to determine D_1, D_2, and D_3 to three decimal places.

Analytical Solution of a Tetrahedron.—The approximate value D'_1 determined by the geometrical method is used for D_1 in determining the value D'_2 of D_2 by Eq. 33*a* and the value D'_3 of D_3 by Eq. 33*b*. Then D'_3 is introduced as the value of D_3 to find the value D''_2 of D_2 by Eq. 33*c*, and finally the correction d_1 is found by

$$D'_2 + \frac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2}\,d_1 = D''_2 + \frac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} \times \frac{D'_3 - D''_2 \cos A_{23}}{D'_3 \cos A_{23} - D''_2}\,d_1 \qquad \text{(36)}$$

Eq. 36, solved for d_1, becomes

$$d_1 = \frac{D'_2 - D''_2}{\dfrac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} \times \dfrac{D'_3 - D''_2 \cos A_{23}}{D'_3 \cos A_{23} - D''_2} - \dfrac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2}} \quad (37)$$

The values obtainable from the foregoing process of calculation are

$$D_1 = D'_1 + d_1 \quad (38a)$$

$$D_2 = D'_2 + \frac{D'_1 - D'_2 \cos A_{12}}{D'_1 \cos A_{12} - D'_2} d_1 \quad (38b)$$

and

$$D_3 = D'_3 + \frac{D'_1 - D'_3 \cos A_{13}}{D'_1 \cos A_{13} - D'_3} d_1 \quad (38c)$$

More accurate values can be found by using the improved value of D_1 to repeat the process.

If the coordinates X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, and Z_3 of points P_1, P_2, and P_3, and the lengths of the sides D_1, D_2, and D_3 are known, coordinates X, Y, and Z of point S are found by solving the equations:

$$D^2_1 = (X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2 \quad (39a)$$

$$D^2_2 = (X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2 \quad (39b)$$

and

$$D^2_3 = (X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2 \quad (39c)$$

from which it follows that

$$2(X_1 - X_2)X + 2(Y_1 - Y_2)Y + 2(Z_1 - Z_2)Z = X^2_1 + Y^2_1 + Z^2_1 - X^2_2 - Y^2_2 - Z^2_2 - D^2_1 + D^2_2 \quad (40a)$$

and

$$2(X_1 - X_3)X + 2(Y_1 - Y_3)Y + 2(Z_1 - Z_3)Z = X^2_1 + Y^2_1 + Z^2_1 - X^2_3 - Y^2_3 - Z^2_3 - D^2_1 + D^2_3 \quad (40b)$$

Thus, X and Y are found in terms of Z in the following form:

$$X = c_1 Z + k_1 \quad (41a)$$

and

$$Y = c_2 Z + k_2 \quad (41b)$$

in which c_1, c_2, k_1, and k_2 are numerical constants. Substituting the equivalents of X and Y into one of the original equations and solving it, two values of Z are obtained which correspond to two positions of point S, one above and one below the base $P_1P_2P_3$.

Refraction of Air.—According to the theory of atmospheric refraction, the total change of direction of the light ray from its position at infinity to point S_1 in the atmosphere, as shown in Fig. 6, is given approximately by

$$C_1 = (g_1 - 1) \tan \zeta_1 \quad (42a)$$

in which C_1 is the angle between the tangent line of the path of light at point S_1 and its original direction; ζ_1 is the angle between the tangent and the plumb line; and g_1 is the refractive index of air at point S_1. Similarly, at another point S_2 on the same path of light,

$$C_2 = (g_2 - 1) \tan \zeta_2 \qquad (42b)$$

The change of direction of the curve from points S_1 to S_2 is equal to

$$C_{12} = (g_2 - 1) \tan \zeta_2 - (g_1 - 1) \tan \zeta_1 \qquad (42c)$$

If the path of the light ray is very flat (as is actually the case), ζ_1 and ζ_2 may be

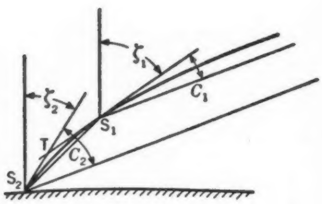

FIG. 6

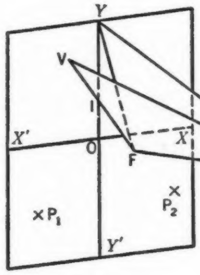

FIG. 7

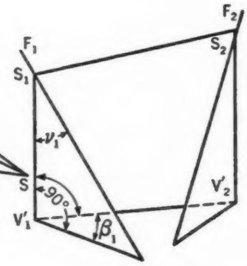

FIG. 8

considered as equal, and, therefore,

$$C_{12} = (g_2 - g_1) \tan \zeta_1 \qquad (42d)$$

In that case it may be reasonably assumed that the angle between the chord S_1S_2 and the tangent at point S_1 is proportional to C_{12}. Consequently,

$$\angle\ \overline{TS_1S_2} = -\rho \tan \zeta_1 \qquad (43)$$

in which ρ is an unknown constant whose value mainly depends on the height of point S_1 above point S_2. The negative sign is used to account for the fact that the zenith distance of the direction of the chord S_2S_1 is smaller than ζ_1, the direction of the tangent TS_1. Since angle TS_1S_2 is always less than the astronomical refraction in numerical value, the value of ρ in minutes of arc is less than 1, if the zenith distance does not exceed 75°.

Problems in Aerial Survey

Camera Constants.—The relation between the rear nodal point and the fiducial axes on the negative plate is essentially as shown in Fig. 2. If the images I_1, I_2, and I_3 of three objects which are at equal angles from each other appear on both fiducial axes, the actual angles θ at the exposure station between the real objects are measured with a theodolite, the lengths OI_1, OI_2, and OI_3 along the fiducial axes on the negative plate can be determined accurately (corrected for distortion if necessary), and the three coordinates of the rear nodal point may be calculated by Eqs. 8 through 12. On account of the errors

of measurement and distortion, the values determined from different sets of data may be slightly different.

When the coordinates of the rear nodal point are exactly known, the space angle at the node subtended by any two image points on the negative plate can be calculated if the coordinate distances of the points can be measured and corrected for distortion.

Position of Exposure Station.—When the coordinate system is so assumed that its (XY)-plane is tangent to the spheroid at a point of given latitude and longitude and the point of tangency is used as origin, the coordinates of a point defined by elevation, direction, and distance from the origin may be calculated by Eqs. 4. The straight distance between two points of known coordinates can be computed easily. If the space angles subtended at the exposure station by three points of known positions can be determined, the position of the exposure station which forms a tetrahedron with the three ground points is determined by the geometrical method or the analytical method (see subsequently in Example 1).

Refraction Correction.—Since the light ray emanating from station S_2 to the camera at station S_1, Fig. 6, follows the same path as the light ray traveling in the opposite direction, the observed direction of point S_2 from point S_1 is deviated an amount given by Eq. 43. To determine this error of refraction, the refractive constant ρ in Eq. 43 must be determined for the exposure station. As an essential requirement, at least four points, P_1, P_2, P_3, and P_4—all with known positions—must be included in one photograph. Let X, Y, and Z be the coordinates of the exposure station S calculated from the space angles A_{12}, A_{13}, and A_{23} and the known positions of points P_1, P_2, and P_3. From the coordinates of S, P_1, P_2, P_3, and P_4 and the center of curvature C of the spheroidal surface, the direction cosines of the lines SP_1, SP_2, SP_3, SP_4, and SC may be found. The angles A_1, A_2, A_3, and A_4 between the plumb line SC and the other four lines can be calculated and may be considered as their observed zenith distances, to which corrections equal to $-\rho \tan A_1$, $-\rho \tan A_2$, $-\rho \tan A_3$, and $-\rho \tan A_4$, respectively, must be added. The correction a_{12} for the space angle A_{12} due to the corrections $-\rho \tan A_1$ and $-\rho \tan A_2$ may be found in terms of ρ by Eq. 25 on the assumption that the dihedral angle between the vertical planes SCP_1 and SCP_2 is not appreciably changed by the effect of refraction. Similarly, the other corrections a_{13}, a_{14}, a_{23}, a_{24}, and a_{34} can all be found in terms of ρ (see subsequently in Example 6).

To account for the corrections a_{12}, a_{13}, and a_{23}, corrections x, y, and z must be added to the calculated coordinates X, Y, and Z of the exposure station S so that Eq. 20 is satisfied. From the three equations thus obtained, the corrections x, y, and z can be found in terms of ρ (see subsequently in Example 7).

Since the corrections of the direction cosines of the lines SP_1, SP_2, and SP_3 can be found in terms of x, y, and z and therefore in terms of ρ by Eqs. 16, and a_{14}, a_{24}, and a_{34} can all be expressed in terms of ρ, the corrections l_4, m_4, and n_4 to be applied to the direction cosines of the line SP_4 may be found in terms of ρ by the three linear equations derived from Eq. 19*b* (see subsequently in Example 8).

The perpendicular distance D_4 from point P_4, which is defined by the known coordinates X_4, Y_4, and Z_4 to line SP_4 (which line, in turn, is defined by a point with coordinates $(X + x)$, $(Y + y)$, and $(Z + z)$ and direction cosines $(L_4 + l_4)$, $(M_4 + m_4)$, and $(N_4 + n_4)$) is found in terms of ρ by Eq. 18. Because of the presence of other errors of measurement, it is not likely that $D^2{}_4 = 0$ can be satisfied by a real value of ρ; but there is always a value of ρ for which $D^2{}_4$ is minimum. Consequently,

$$\frac{d(D^2{}_4)}{d\rho} = 0 \quad \text{(44a)}$$

may be used to determine ρ. Similarly if several other points P_5, P_6, and P_7 (also of known positions) appear in the same photograph, the value of ρ, determined from

$$\frac{d(D^2{}_4 + D^2{}_5 + D^2{}_6 + D^2{}_7)}{d\rho} = 0 \quad \text{(44b)}$$

should be more reliable.

Equation to Determine ρ.—Let L_4, M_4, and N_4 be the direction cosines of the line which makes angles A_{14} and A_{24} with the lines SP_1 and SP_2. When the coordinates of point S are changed to $(X + x)$, $(Y + y)$, and $(Z + z)$ and the angles A_{14} and A_{24} are changed to $(A_{14} + a_{14})$ and $(A_{24} + a_{24})$ to account for the effect of refraction, the corrections x, y, z, l_4, m_4, and n_4 may all be found in terms of ρ as illustrated subsequently by Examples 7 and 8. Since —$D^2{}_4 = (X + x - X_4)^2 + (Y + y - Y_4)^2 + (Z + z - Z_4)^2 - [(L_4 + l_4) \times (X + x - X_4) + (M_4 + m_4)(Y + y - Y_4) + (N_4 + n_4)(Z + z - Z_4)]^2$— is an expression involving the fourth power of ρ, the expression:

$$\frac{dD^2{}_4}{d\rho} = 0 \quad \text{(45)}$$

gives an equation of the form:

$$a + a_1\rho + a_2\rho^2 + a_3\rho^3 = 0 \quad \text{(46)}$$

the solution of which gives the value of ρ.

Orientation of Negative Plates.—As shown in Fig. 7, point S is the rear nodal point of the camera lens; X, X', Y, and Y' are the fiducial marks on the negative plate, point F is the plate intercept of a perpendicular from point S, and point V is the piercing point, on the negative plate, of the plumb line SV. Since the space angles at point S subtended by points Y, P_1, and P_2 and the equations of the lines SP_1 and SP_2 are determinable, the equation of line SV is computed by Eqs. 26. Similarly, the equation of SF may be found also. The equation of the plumb line SV is determined from the calculated coordinates of point S and the coordinates of the center of curvature of the spheroidal surface.

The angles between the three lines SY, SF, and SV are found from their equations; and the angle YFV is found from these angles by Eq. 24. Angle FYY′ can be found from the coordinates of points Y and F; and, as in the case shown, the angle between the lines FV and YY′, is equal to

$$\angle\,\overline{FIY'} = \angle\,\overline{FYY'} + \angle\,\overline{YFV} \quad \text{(47)}$$

Since the equation of the plane of SV and SF can be determined from the equations of the two lines, and the equation of the plane of the meridian through point S is found from Eq. 5, the azimuth of line SF is found from the dihedral angle between these two planes.

The coordinates of point F, the azimuth of line SF, and angles VSF and FIY′ completely define the orientation of the negative plate at the instant of exposure.

As shown in Fig. 8, points S_1 and S_2 are two exposure stations and V'_1 and V'_2 represent the feet of the plumb lines on the spheroidal surface. The elevations V'_1S_1 and V'_2S_2 can be found from the calculated coordinates of points S_1 and S_2 by Eq. 1. The azimuth of the line S_1S_2 is found from the dihedral angle between the meridian plane of point S_1 and the vertical plane containing points S_1 and S_2.

Since the equations of lines S_1S_2 and $S_1V'_1$ and the plate perpendicular S_1F_1 can be found, it is possible to compute the angle ν_1, Fig. 8, which may be considered as the vertical angle of the line of sight of the camera, and the angle $V'_1S_1S_2$. The horizontal angle β_1 is determined from the three space angles at point S_1 by Eq. 24. Similarly, the vertical and horizontal angles of the plate perpendicular S_2F_2 at point S_2 are found. These angles, together with the angles of orientation of the two negative plates, completely define their relative position in the air.

Position of Photographed Point.—The space angles at the exposure station S_1 made by a point P_n of unknown position with known points P_1, P_2, and P_3 can be determined from their images on the negative plate by Eqs. 7 and they can be corrected for atmospheric refraction by Eq. 25, when the position of station S_1 and the refractive constant ρ are known. From these space angles the direction cosines of the line S_1P_n may be determined from Eqs. 26 or 32 according to the number of known points available.

Similarly, if point P_n is also photographed at another exposure station S_2 and the equation of the line S_2P_n can be determined, the accuracy of all the data may be tested by computing the perpendicular distance between the two lines S_1P_n and S_2P_n with Eq. 21*a*. If this distance is reasonably small, its midpoint may be considered as the probable position of point P_n and its coordinates may be determined by Eqs. 23.

Four or More Points of Known Positions in One Photograph.—In this case more data are available than necessary and the method of least squares is used to determine the coordinates of the exposure station S.

The coordinates of point S are first calculated from the space angles of points P_1, P_2, and P_3. Then the lengths and direction cosines of lines SP_1, SP_2, SP_3, and SP_4 are calculated; and the small corrections a_{12}, a_{13}, and a_{23} of the space angles A_{12}, A_{13}, and A_{23} due to corrections x, y, and z which must be added to the calculated coordinates of point S are found from Eq. 20.

Let the values of angles A_{14}, A_{24}, and A_{34}, calculated from the direction cosines of the four lines, be denoted by I_{14}, I_{24}, and I_{34} and their increases in terms of x, y, and z, found by Eq. 20, be denoted by i_{14}, i_{24}, and i_{34}, respectively. The corrections a_{14}, a_{24}, and a_{34} when added to the observed values A_{14}, A_{24},

and A_{34}, respectively, must yield

$$I_{14} + i_{14} = A_{14} + a_{14} \qquad (48a)$$

$$I_{24} + i_{24} = A_{24} + a_{24} \qquad (48b)$$

and

$$I_{34} + i_{34} = A_{34} + a_{34} \qquad (48c)$$

Therefore,

$$a_{14} = I_{14} - A_{14} + i_{14} \qquad (49a)$$

$$a_{24} = I_{24} - A_{24} + i_{24} \qquad (49b)$$

and

$$a_{34} = I_{34} - A_{34} + i_{34} \qquad (49c)$$

which give the angular corrections a_{14}, a_{24}, and a_{34}, also, as linear functions of x, y, and z.

As it is possible to restrain the angular corrections if the sum Q, defined by the condition that—

$$Q = W_{12}(a_{12})^2 + W_{13}(a_{13})^2 + W_{23}(a_{23})^2 + W_{14}(a_{14})^2 + W_{24}(a_{24})^2 + W_{34}(a_{34})^2 \qquad (50)$$

—shall be a minimum, the values of x, y, and z can be determined from the three normal equations derived from $\frac{dQ}{dx} = 0$, $\frac{dQ}{dy} = 0$, and $\frac{dQ}{dz} = 0$, respectively (see subsequently in Example 5). In Eq. 50, W is the weight of a corresponding observed value. The foregoing method can be extended to other cases involving five or six ground points of known positions.

Four or More Points of Known Positions and One or Several Points of Unknown Positions Common to Two Aerial Photographs.—Let the positions of four ground points P_1, P_2, P_3, and P_4 be given, with that of P_5 undetermined, and let the available data be composed of the space angles A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}, A_{15}, A_{25}, and A_{35} and A'_{12}, A'_{13}, A'_{14}, A'_{23}, A'_{24}, A'_{34}, A'_{15}, A'_{25}, and A'_{35} at the exposure stations S and S′, respectively. In the solution, the coordinates of points S and S′ are first computed from the given coordinates of points P_1, P_2, and P_3 by the use of the space angles at S and S′ upon these points. Then, corrections to these coordinates x, y, z, x', y', and z', respectively, and other corrections are computed as explained in the following text.

For the points of known positions, corrections a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}, a'_{12}, a'_{13}, a'_{23}, a'_{14}, a'_{24}, and a'_{34} of their space angles may be expressed in terms of x, y, and z and x', y', and z', respectively, by Eq. 20. For point P_5 (position unknown), the direction cosines of lines SP_5 and $S'P_5$ are found from their space angles fixed by three points of known positions such as P_1, P_2, and P_3 by Eqs. 32; and their corrections l_5, m_5, n_5, l'_5, m'_5, and n'_5 are also to be determined. The correction a_{15} may be expressed in terms of l_1, m_1, n_1, l_5, m_5, and n_5 by Eq. 19b; l_1, m_1, and n_1 can be obtained in terms of x, y, and z; and n_5 can be found in terms of l_5 and m_5 by Eq. 17. Therefore, a_{15} can be expressed in terms of x, y, z, l_5, and m_5. Similarly, a_{25} may be expressed in terms of the same five corrections. The correction a_{35} should be determined from a_{12}, a_{13}, a_{23}, a_{15}, and a_{25} by Eq. 30 and therefore may also be expressed in terms of the

foregoing five corrections. Likewise a'_{15}, a'_{25}, and a'_{35} may be expressed in terms of x', y', z', l'_5, and m'_5.

To make the two lines SP_5 and $S'P_5$ intersect, however, corrections x, y, z, x', y', z', l_5, m_5, l'_5, and m'_5 must satisfy a conditional equation based on Eq. 22 in which n_5 and n'_5 are replaced by their equivalents derived from Eqs. 16. By this condition of intersection, one of the corrections such as m'_5 can be found in terms of the other nine.

The foregoing analysis shows that the basic corrections in this case are x, y, z, x', y', z', l_5, m_5, and l'_5 which may be determined from a set of normal equations if the sum of the weighted squares of the corrections of the space angles is a minimum. The method is general and applicable to the case with $(3 + q)$-points of known positions and u points of unknown positions; but the number of basic corrections, being equal to $3\ (2 + u)$ is dependent on u only. The method can also be modified to include such data as A_{45} and A'_{45}. If the refractive constant ρ is known, the space angles should first be corrected for refraction by Eq. 25, and then they should be used as the data to start the aforementioned method of analysis.

If (because of some special arrangement to be explained subsequently) the position of the exposure station S has been determined previously, the corrections x, y, and z may be considered equal to zero. The number of the basic corrections is then decreased by 3, thus decreasing the work of numerical calculation greatly.

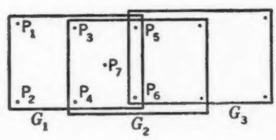

FIG. 9

Analysis of Partly Overlapping Aerial Survey.—Fig. 9 represents an aerial survey with a 60% overlap between photographs G_1, G_2, and G_3. The method of successive approximation can be applied to cases of this kind. It is assumed at the beginning that: (1) At least three points of known positions, such as P_1, P_3, and P_4, are available for the determination of the exposure station S_1 at which photograph G_1 is taken; (2) in photograph G_2 only two points of known positions such as P_3 and P_4 can be identified; and (3) each of the succeeding photographs contains at least two well-defined points in common with two previous photographs. It is also assumed that: (4) The angles of tilt are small and (5) the altimeter records are reasonably reliable. Under these assumed conditions, the exposure station may be represented approximately, in plan, by the principal point, and in elevation by the altimeter reading. If the principal point of a vertical photograph can be identified in a previous photograph by its surrounding topographic details, its position in plan is estimated or determined by the radial-line method of plotting.

From the known coordinates of points P_3 and P_4, and from the estimated coordinates of the exposure station S_2 at which photograph G_2 is taken, the direction cosines of the lines S_2P_3 and S_2P_4 and their including angle I''_{34} may be calculated. The change i''_{34} of I''_{34} due to corrections x_2, y_2, and z_2 to be added to the estimated coordinates X_2, Y_2, and Z_2 of point S_2 may be found by Eq. 20. Since the sum $(I''_{34} + i''_{34})$ must be equal to the observed value A''_{34} of the space angle at point S_2,

$$i''_{34} = A''_{34} - I''_{34} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

which is a linear equation in x_2, y_2, and z_2. If it is believed that x_2 and y_2 are relatively small because X_2 and Y_2 are more reliably determined than Z_2, Eq. 51 can be used to find z_2 by omitting the terms involving x_2 and y_2. In that case, the calculation should be begun again with X_2, Y_2, and $(Z_2 + z_2)$ as the initial values. It is essential that good initial values be obtained by a simple method so that the entire process of calculation can be shortened.

The direction cosines of line S_2P_5 may be calculated from the computed values of the direction cosines of lines S_2P_3 and S_2P_4 and the space angles A''_{35} and A''_{45} by Eqs. 26. The changes in the direction cosines of lines S_2P_3 and S_2P_4 due to the corrections x_2, y_2, and z_2, and the corrections l''_5, m''_5, and n''_5 to be applied to the calculated values of the direction cosines of line S_2P_5 must not alter the values of the space angles A''_{35} and A''_{45}. Therefore,

$$L''_5 l''_3 + L''_3 l''_5 + M''_5 m''_3 + M''_3 m''_5 + N''_5 n''_3 + N''_3 n''_5 = 0 \quad (52a)$$

$$L''_5 l''_4 + L''_4 l''_5 + M''_5 m''_4 + M''_4 m''_5 + N''_5 n''_4 + N''_4 n''_5 = 0 \quad (52b)$$

and

$$L''_5 l''_5 + M''_5 m''_5 + N''_5 n''_5 = 0 \quad (52c)$$

in which all the capital letters denote numerical constants determined from known or assumed values. Corrections l''_3, m''_3, n''_3, l''_4, m''_4, and n''_4 can be found in terms of x_2, y_2, and z_2 by Eqs. 16; therefore corrections l''_5, m''_5, and n''_5 can also be obtained in terms of x_2, y_2, and z_2 by Eqs. 16. The line S_2P_5 must intersect line S_1P_5, however; and the equation of the latter has been determined completely in the analysis of the previous survey. Hence, the equation of intersection based on Eq. 22 involves only x_2, y_2, z_2, l''_5, m''_5, and n''_5 as the unknowns; and, as the latter three corrections may be found in terms of the corrections of coordinates, the final result is again a linear equation in x_2, y_2, and z_2. Similarly, the intersection of the lines S_1P_6 and S_2P_6 gives another equation in x_2, y_2, and z_2. These three equations are used to determine the values of the three corrections.

As the errors of the initial values X_2, Y_2, and Z_2 may be very large, the corrected values $(X_2 + x_2)$, $(Y_2 + y_2)$, and $(Z_2 + z_2)$ are usually only better approximations. With these revised values of the coordinates of point S_2 as the new base for starting, the foregoing procedure is repeated (see subsequently in Example 9).

Since there may be other well-defined points such as P_7 common to photographs G_1 and G_2, and since from each common point a linear equation in x_2, y_2, and z_2 can be derived, more equations may be obtained than are required for the determination of the three unknowns. In such a case, in the second course of approximation, the method of least squares is used.

After the coordinates of point S_2 are thus calculated, the equations of the lines from point S_2 to all points in photograph G_2 may be computed, and the positions of the points common to photographs G_1 and G_2 may be determined by solving Eqs. 23.

The same method of calculation may be used for obtaining the positions of the succeeding exposure stations and points, until the survey ends or closes on another point of known position.

Aerial Triangulation.—Present practice in aerial surveying is mainly based on the use of partly overlapping (generally 60%) exposures as shown in Fig. 9. The method of computing such a survey is not straightforward although results may be obtained which are as reliable as the angular data.

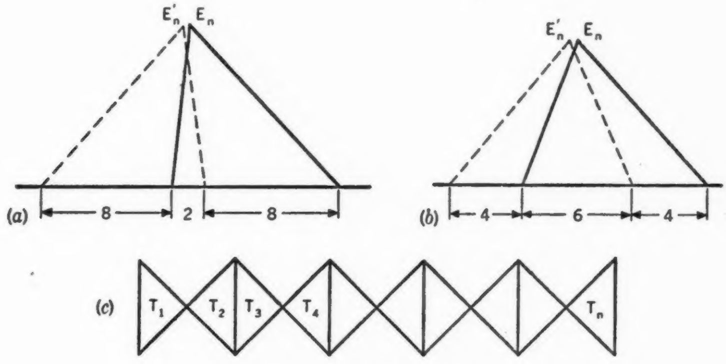

Fig. 10.—Aerial Triangulation

A fully duplicated, and also partly overlapping, aerial survey may be made with two cameras so arranged that their fields of view have a longitudinal overlap of 20% as shown in Fig. 10(*a*) or of 60% as shown in Fig. 10(*b*). When cameras arranged in such a combination are operated simultaneously by electric control so that the exposures of each camera have a 60% overlap of their own, each photograph of one camera will be nearly duplicated by some photograph of another.

Let a series of triangles T_1, T_2, $\cdots$, T_n on a strip of ground surface (see Fig. 10(*c*)) be so spaced that adjacent pairs will appear together in the same photograph; and let the simultaneous exposures of the two cameras at the *j*th exposure station be denoted by E_j and E'_j. If triangle T_1 is composed of points of known positions and may be used to determine the position of the exposure

TABLE 1*A*.—Schedule of Sequence in the Use of Triangles in Aerial Triangulation Computations; 25% Overlap

Exposures	Backsights	Foresights
E_1	T_1	T_2
E'_3	T_1	T_2
E_2	T_2	T_3
E'_4	T_2	T_3
....	....	....
E_n	T_n	T_{n+1}
E'_{n+2}	T_n	T_{n+1}

TABLE 1*B*.—Schedule of Sequence in the Use of Triangles in Aerial Triangulation Computations; 60% Overlap

Exposures	Backsights	Foresights
E_1	T_1	T_2
E'_2	T_1	T_2
E_2	T_2	T_3
E'_3	T_2	T_3
....	....	....
E_n	T_n	T_{n+1}
E'_{n+1}	T_n	T_{n+1}

station, the two sets of exposures can be compiled as in Tables 1*A* and 1*B*, showing the sequence in which calculations are made. As in the case of a ground survey, the sights used to determine the positions of the instrument are called the "backsights," and those used to determine the points sighted are called the "foresights." In either case the backsights of triangle T_1 are used to find the positions of the exposure stations S_1 and S'_3 or S_1 and S'_2 and the foresights from two stations of known positions are used to find the vertexes of triangle T_2. This process of calculation is continued until the series of triangles ends on another point of known position.

As the relative position of the exposure stations S_n and S'_n is identical to that of the two cameras in the airplane, when their positions are determined independently, additional checks on the accuracy of the data and the numerical calculations are provided by the spacing of the cameras.

Error of Closure of Aerial Triangulation.—When an aerial survey begins from a ground triangle composed of points of known positions and ends on another known ground point, the coordinates X_a, Y_a, and Z_a of the terminal point determined by the aerial survey should agree with the more reliable values X_g, Y_g, and Z_g determined by the ground survey. The errors in three directions are: $e_x = X_a - X_g$, $e_y = Y_a - Y_g$, and $e_z = Z_a - Z_g$, respectively, and the error of closure is equal to $e_c = \sqrt{e^2_x + e^2_y + e^2_z}$. Let the distance between one of the starting points and the terminal point be D. The ratio $\frac{e}{D}$, reduced to the form $\frac{1}{N}$, will serve as a measure of the accuracy of the survey.

If it is assumed that the errors e_x, e_y, and e_z are accumulated equally from all parts of the survey, the corrections to be added to the calculated coordinates of an intermediate point should be: $\frac{-e_x D_i}{D}$, $\frac{-e_y D_i}{D}$, and $\frac{-e_z D_i}{D}$, respectively, in which D_i is the distance of the point from the starting point.

TABLE 2.—COORDINATES OF GROUND POINTS, IN METERS
(1 Meter = 3.28083 Ft)

Point	*X*	*Y*	*Z*
P_1......	2,578.45	3,876.38	68.49
P_2......	5,769.38	1,643.21	135.27
P_3......	6,043.75	5,872.15	146.73
P_4......	2,385.00	5,934.00	76.30

ILLUSTRATIVE EXAMPLES

Data.—The data in Table 2 are assumed for the purpose of making sample calculations that will illustrate the general features of the methods and offer some indication as to the accuracy of the results to be expected. The positions of two exposure stations are also assumed so that the field angles to the various ground points are, for station S_1,

Angle	Value
$P_1S_1P_2$	67° 31′
$P_1S_1P_3$	69° 47′
$P_2S_1P_3$	75° 38′
$P_1S_1P_4$	29° 08.011′
$P_2S_1P_4$	89° 53.239′
$P_3S_1P_4$	55° 47.648′

and, for station S_2,

Angle	Value
$P_1S_2P_2$	44° 04.508′
$P_1S_2P_3$	62° 05.500′
$P_1S_2P_4$	29° 24.033′
$P_2S_2P_3$	49° 10.383′
$P_2S_2P_4$	68° 52.333′

Example 1. To Find the Coordinates X_1, Y_1, and Z_1 of Exposure Station S_1.—The lengths D_1, D_2, and D_3 of lines S_1P_1, S_1P_2, and S_1P_3 must first be determined from the equations:

$$D^2_1 - 2(0.3824147)\, D_1\, D_2 + D^2_2 - 15{,}173{,}542 = 0 \quad \text{.......(53}a\text{)}$$

$$D^2_1 - 2(0.3455712)\, D_1\, D_3 + D^2_3 - 15{,}997{,}524 = 0 \quad \text{.......(53}b\text{)}$$

and

$$D^2_2 - 2(0.2481263)\, D_2\, D_3 + D^2_3 - 17{,}959{,}339 = 0 \quad \text{.......(53}c\text{)}$$

The results are summarized in Table 3. The final values are accurate to the

TABLE 3.—COMPUTATION OF LENGTHS D BY EQS. 53 (IN METERS)

Trial No.	D'_1		D'_2		D'_3		D''_2	d_1
	Assumed (1)	Corrected (2)	Computed (3)	Corrected (4)	Computed (5)	Corrected (6)	Computed (7)	Computed (8)
1...	+3,505	+3,544.60	+3,504.87	+3,465.27	+3,487.09	+3,447.09	+3,424.18	+39.6
2...	+3,545	+3,543.957	+3,463.818	+3,464.918	+3,445.691	+3,446.791	+3,466.005	− 1.043
3...	+3,543.957	+3,543.963	+3,464.916	+3,464.910	+3,446.796	+3,446.791	+3,464.904	+ 0.0057

third decimal place. The coordinates of point S_1 (in meters) are then found to be: $X_1 = 4{,}953.549$, $Y_1 = 3{,}827.388$, and $Z_1 = 2{,}698.353$.

Example 2. To Find the Elevation of Station S_1.—Let the radius of curvature of the spheroidal surface be $R = 6{,}363{,}200$ meters (1 meter = 3.28083 ft). It is obvious that the third term in Eq. 1 can be neglected in this case and that the elevation P_fS_1, therefore, is $2{,}698.353 + \dfrac{19{,}593{,}000}{6{,}365{,}900}$ = 2,701.431 m.

TABLE 4.—COMPUTATION OF DIRECTION COSINES

Line	L	M	N
S_1P_f....	−0.0007781	−0.0006012	−0.9999995
S_1P_1.....	−0.6701815	+0.0138241	−0.7420682
S_1P_2.....	+0.2354551	−0.6303707	−0.7397255
S_1P_3.....	+0.3162945	+0.5932365	−0.7402895
S_1P_4.....	−0.6069250	+0.4977735	−0.6195673

Example 3. To Find the Direction Cosines of Lines S_1P_f, S_1P_1, S_1P_2, S_1P_3, S_1P_4, *and the Included Angles.*—The direction cosines of the plumb line S_1P_f are determined from the coordinates of station S_1, and the center of curvature ($X = 0$, $Y = 0$, and $Z = -\ 6{,}363{,}200$). Similarly, the direction cosines of the other four lines may be found from the known coordinates (see Table 4). From the direction cosines the following space angles may be com-

puted by Eq. 19*a*:

Angle	Value
$P_fS_1P_1$	42° 02.893′
$P_fS_1P_2$	42° 16.518′
$P_fS_1P_3$	42° 17.720′
$P_fS_1P_4$	51° 42.172′

and, as measured to P_4,

Angle	Value
$P_1S_1P_4$	29° 08.681′
$P_2S_1P_4$	89° 54.417′
$P_3S_1P_4$	55° 48.389′

Example 4. To Find the Direction Cosines of Line S_1P_4 *from the Direction Cosines of Lines* S_1P_1, S_1P_2, *and* S_1P_3.—The direction cosines of line S_1P_4 found from the calculated values of angles $P_1S_1P_4$, $P_2S_1P_4$, and $P_3S_1P_4$ by Eqs. 32 are: $L_4 = -0.6069253$, $M_4 = +0.4977740$, and $N_4 = -0.6195673$.

TABLE 5.—DIFFERENCES BETWEEN THE ASSUMED AND COMPUTED VALUES OF ANGLES AT STATION S_1

Angle	Assumed	Computed	Error
$P_1S_1P_4$	29° 08.011′	29° 08.681′	0° 00.670′
$P_2S_1P_4$	89° 53.239′	89° 54.417′	0° 01.178′
$P_3S_1P_4$	55° 47.648′	55° 48.389′	0° 00.741′

Example 5. To Find the Most Probable Position of S_1.—Table 5 demonstrates the differences between the assumed and the calculated values of the angles $P_1S_1P_4$, $P_2S_1P_4$, and $P_3S_1P_4$. If these errors are to be distributed to the six space angles at station S_1 by introducing corrections x_1, y_1, and z_1 to the calculated coordinates of station S_1, it may be found from Eq. 20 that

$$a_{12} = -0.30007\,x_1 - 0.39364\,y_1 - 0.97174\,z_1 \qquad (54a)$$

$$a_{13} = -0.26216\,x_1 + 0.40511\,y_1 - 1.0170\,z_1 \qquad (54b)$$

and

$$a_{23} = 0.42573\,x_1 - 0.03279\,y_1 - 1.1427\,z_1 \qquad (54c)$$

and Eqs. 20 and 49 will yield

$$a_{14} = 0.670 - 0.27669\,x_1 + 0.26535\,y_1 - 0.27829\,z_1 \qquad (55a)$$

$$a_{24} = 1.178 - 0.41046\,x_1 - 0.01784\,y_1 - 1.2136\,z_1 \qquad (55b)$$

and

$$a_{34} = 0.741 - 0.30059\,x_1 + 0.50608\,y_1 - 0.63048\,z_1 \qquad (55c)$$

The angular corrections are all reduced to minutes of arc by multiplying the results by 3,437.7, which is the number of minutes in one radian. The normal equations derived from these corrections, with equal weights, are

$$0.6786\,x_1 - 0.2203\,y_1 + 0.8364\,z_1 = 0.8916 \qquad (56a)$$

$$-0.2203\,x_1 + 0.6470\,y_1 - 0.3633\,z_1 = -0.5317 \qquad (56b)$$

and

$$0.8364\,x_1 - 0.3633\,y_1 + 5.2323\,z_1 = 2.0830 \qquad (56c)$$

from which $x_1 = 0.9105$, $y_1 = -0.3851$, and $z_1 = 0.2258$. These are the corrections to be added algebraically to the coordinates of S_1, respectively.

Example 6. To Find the Corrections of the Space Angles for Atmospheric Refraction.—By using the calculated values of the space angles between the lines S_1P_f, S_1P_1, S_1P_2, S_1P_3, and S_1P_4, Eq. 25 yields: $a_{12} = -1.3368\,\rho$, $a_{13} = -1.3948\,\rho$, $a_{23} = -1.5523\,\rho$, $a_{14} = -0.5873\,\rho$, $a_{24} = -2.0283\,\rho$, and $a_{34} = -1.0970\,\rho$—in which ρ is in minutes.

Example 7. To Find the Corrections of the Coordinates of S_1 *in Terms of* ρ.—By equating the errors of refraction in angles P_1SP_2, $P_1S_1P_3$, and $P_2S_1P_3$ to those due to the corrections x_1, y_1, and z_1 to be added to the calculated coordinates of station S_1, three equations (in meters)—

$$0.30007\,x_1 + 0.39364\,y_1 + 0.97174\,z_1 = 1.3368\,\rho \tag{57a}$$

$$0.26216\,x_1 - 0.40511\,y_1 + 1.0170\,z_1 = 1.3945\,\rho \tag{57b}$$

and

$$-0.42573\,x_1 + 0.03279\,y_1 + 1.1427\,z_1 = 1.5523\,\rho \tag{57c}$$

—may be obtained which give: $x_1 = 0.0231\,\rho$, $y_1 = 0.0042\,\rho$, and $z_1 = 1.3669\,\rho$.

Example 8. To Find the Corrections of the Direction Cosines in Terms of ρ.—From the values of x_1, y_1, and z_1 found in Example 7 and from other known values, by Eqs. 16, the corrections of the direction cosines of S_1P_1, S_1P_2, and S_1P_3 may be determined as shown in Table 6 in which l_4, m_4, and n_4 are found from the three linear equations derived from Eq. 19*b*.

TABLE 6.—CORRECTIONS FOR DIRECTION COSINES, EXAMPLE 8

Line	l	m	n
S_1P_1.....	$+0.0001882\,\rho$	$-0.0000057\,\rho$	$-0.0001701\,\rho$
S_1P_2.....	$-0.0000752\,\rho$	$+0.0001822\,\rho$	$-0.0001792\,\rho$
S_1P_3.....	$-0.0000985\,\rho$	$-0.0001737\,\rho$	$+0.0001813\,\rho$
S_1P_4.....	$+0.0001706\,\rho$	$-0.0001457\,\rho$	$-0.0002847\,\rho$

Example 9. To Find the Coordinates of Station S_2 *from Space Angles, and the Known Positions of Points* P_1 *and* P_2.—The initial values assumed for the coordinates of station S_2 are: $X_2 = 4{,}400$, $Y_2 = 6{,}210$, and $Z_2 = 3{,}160$. With

TABLE 7.—COORDINATES OF STATION S_2 COMPUTED FROM SPACE ANGLES

Trial No.	X_2			Y_2			Z_2		
	Assumed (1)	Corrections (2)	Corrected (3)	Assumed (4)	Corrections (5)	Corrected (6)	Assumed (7)	Corrections (8)	Corrected (9)
1.....	+4,400	−58.5	+4,341.5	+6,210	−69.7	+6,140.3	+3,160	−19.8	+3,140.2
2.....	+4,341	+17.6	+4,358.6	+6,140	+10.4	+6,150.4	+3,140	− 8.2	+3,131.8
True value..	....	....	+4,358.00	....	....	+6,150.00	....	....	+3,132.00

these and other known values the solution of Eq. 51 yields: $536.9\,x_2 - 2{,}160.7\,y_2 - 1{,}499.2\,z_2 = 150{,}214$, as a correction for angle $P_1S_2P_2$. Similarly, Eq. 22 yields

$$0.1691\,x_2 + 0.1015\,y_2 + 0.5552\,z_2 = -28.02$$

and

$$0.2796\,x_2 - 0.5559\,y_2 - 0.4415\,z_2 = 31.12.$$

for making lines S_2P_3 and S_2P_4 intersect the lines S_1P_3 and S_1P_4, respectively. The solution of these three equations gives: $x_2 = -58.5$, $y_2 = -69.7$, and $z_2 = -19.8$. The foregoing results and those of a further process of approximation are shown in Table 7 which indicates a steady convergence to the true values.

Acknowledgment

In the final editing stage, when publication promised to be greatly delayed by difficulties of long-distance communication, Mr. Underwood very kindly served as the writer's "proxy," thus advancing the release of this paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1950

PRESIDENT

ERNEST E. HOWARD

VICE-PRESIDENTS

Term expires January, 1951:

HENRY J. SHERMAN
ROBERT B. BROOKS

Term expires January, 1952:

FRED C. SCOBEY
ALBERT HAERTLEIN

DIRECTORS

Term expires January, 1951:

WILLIAM M. GRIFFIN
KIRBY SMITH
JOEL D. JUSTIN
FRANCIS S. FRIEL
WEBSTER L. BENHAM
C. GLENN CAPPEL

Term expires January, 1952:

WALDO G. BOWMAN
MORRIS GOODKIND
HAROLD L. BLAKESLEE
PAUL L. HOLLAND
EDMUND FRIEDMAN
S. T. HARDING

Term expires January, 1953:

OTTO HOLDEN
FRANK L. WEAVER
GORDON H. BUTLER
LOUIS R. HOWSON
G. BROOKS EARNEST
WALTER J. RYAN
GEORGE W. LAMB

PAST-PRESIDENTS

Members of the Board

R. E. DOUGHERTY

FRANKLIN THOMAS

TREASURER

CHARLES E. TROUT

EXECUTIVE SECRETARY

WILLIAM N. CAREY

ASSISTANT TREASURER

GEORGE W. BURPEE

ASSISTANT SECRETARY

E. L. CHANDLER

PROCEEDINGS OF THE SOCIETY

SYDNEY WILMOT
Manager of Technical Publications

HAROLD T. LARSEN
Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

WALDO G. BOWMAN

FRANCIS S. FRIEL
S. T. HARDING
OTTO HOLDEN
LOUIS R. HOWSON
KIRBY SMITH